Math 1522 - Exam 4 Study Guide

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Contents

Summary and Disclaimer

This is a study guide for the first exam for math 1522 at the University of New Mexico (Calculus II). The exam covers sections 11.8 through 11.11 of Stewart's Calculus, as well as basic complex numbers (which can be found in Appendix H of the same text). As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material from a calculus 1 course and the material previously covered in calculus 2. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

If you are not in my class, I cannot guarantee how much these notes will help you. With that said, if your TA or instructor has shared these with you, then you will most likely get some use out of them.

Methods and Techniques

The primary focus of this exam is on power series. For this exam, there are four power series that you should have memorized:

Common Power Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Since we mainly use Taylor series in this course, you should know how they are defined:

Taylor Series

The Taylor series for a function $f(x)$ centered at $x = a$ has the formula

$$\sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

It is also nice to know what a Taylor polynomial is, since this often comes up:

Taylor Polynomials

The nth Taylor polynomial of a function $f(x)$ centered at $x = a$ is

$$p_n(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

You will notice that this is just the first n terms of the Taylor series (we just had to use k as our variable since n was already in use). This means that it is an estimation of the regular Taylor series, which means that it has some error attached to it. To find the error, we have Taylor's Estimation Theorem:

Taylor's Estimation Theorem

The difference between $p_n(x)$ (centered at a) and $f(x)$ on an interval $[b, c]$ (equivalently for $b \leq x \leq c$) is given by

$$|p_n(x) - f(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

Where M is the maximum value of $|f^{(n+1)}(x)|$ on the interval $[b, c]$.
It is often useful to note that $|x - a|$ is less than the distance from a to the furthest endpoint of the interval (be it b or c).

Finally, depending on your instructor you may need to know some facts about complex numbers.

Notation of Complex Numbers

There are two ways of writing a complex number z. The first way is called standard form, and we write $z = x + iy$, where x and y are real numbers.
The second way to write a complex number is called polar form, and we write $z = re^{i\theta}$, where r is a positive real number (or 0), and θ is an angle between 0 and 2π.

The reason why this second form works is because of Euler's formula. This is the second major thing about complex numbers that you should know:

Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Although it has no explicit name, the identity $e^{(\theta+2\pi)i} = e^{\theta i}$, which can be derived from the above, is also rather useful.

Worked Examples

We will now work through some examples.

Example: Find the power series for $f(x) = \ln(1 + x^2)$.
We begin by taking the derivative of $\ln(1 + x^2)$. This gives us (by the chain rule),

$$f'(x) = \frac{2x}{1 + x^2}.$$

This is a geometric series, since we can write it as

$$f'(x) = 2x\frac{1}{1 - (-x^2)}$$

so

$$f'(x) = 2x\left(1 - x^2 + x^4 - x^6 + \ldots\right).$$

Distribution then gives us that

$$f'(x) = 2x - 2x^3 + 2x^5 - 2x^7 + \ldots.$$

Integrating this gives us

$$f(x) = C + \frac{2x}{2} - \frac{2x^4}{4} + \frac{2x^6}{6} - \frac{2x^8}{8} + \ldots.$$

We simplify this down to

$$f(x) = C + x - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots,$$

and plug in $x = 0$ to get $C = \ln(1 + 0)) = 0$. So,

$$f(x) = x - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots.$$

Example: Determine the value of

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{2n} n!}$$

We have that $3^{2n} = 9^n$, so, we can rewrite the sum as

$$\sum_{n=0}^{\infty} \frac{2^n}{9^n n!} = \sum_{n=0}^{\infty} \frac{\frac{2^n}{9^n}}{n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{2}{9}\right)^n}{n!}.$$

But this is just the power series of e^x with $\frac{2}{9}$ where the x should be. So,

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{2n} n!} = e^{\frac{2}{9}}.$$

Example: Consider the Taylor series of $\cos(x)$ centered at $a = 0$. Find $p_2(x)$, and determine the error on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Recall that the Taylor series for $\cos(x)$ is given by

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$$

So, $p_2(x) = 1 - \dfrac{x^2}{2}$, since $2! = 2$.
For the second part of the question, we have two ways of doing this. The first is the alternating series remainder theorem, and the second is Taylor's Estimation Theorem. First, we will handle the alternating series remainder theorem, since this is easier.
For the alternating series estimation theorem, we know that the value of the estimation is less than the value of the next term. So, if E is the error in the estimation,

$$E \leq \left|\frac{x^4}{4!}\right| = \frac{|x^4|}{4!} = \frac{|x|^4}{24}.$$

So, we find the maximum value of $|x|$ on our interval. And since the endpoints are where this is greatest, and the absolute value is equal to $\frac{\pi}{2}$ at the endpoints, we have that

$$|x| \leq \frac{pi}{2}.$$

So,

$$E \leq \frac{\frac{\pi^4}{2^4}}{24} = \frac{\pi^4}{2^4 \cdot 24} = \frac{\pi^4}{384}.$$

For Taylor's Estimation Theorem, we are using the formula

$$E \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

where E is the error and M is the maximum value of the $(n+1)$th derivative of f on the interval.
So, we first find the maximum value of $|x - a|$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since $a = 0$, this is just the maximum value of $|x|$, which we found above to be $\frac{\pi}{2}$.
Next, we need to find the third derivative of $\cos(x)$, since $n = 2$ and we need to find

the $(n+1)$th derivative. This is just $\sin(x)$. So, we need to find the maximum value of $|\sin(x)|$ on the indicated interval as well, since this is M in our formula. However, $\sin(x)$ is always between -1 and 1, and it attains these values at the endpoints of our interval. So, $M = 1$. Plugging these things into our formula gives us

$$E \leq \frac{\frac{\pi^4}{2^4}}{24} = \frac{\pi^4}{2^4 \cdot 24} = \frac{\pi^4}{384},$$

just as in our previous error calculation.

Practice Problems

These practice problems are separate from the unsolved problems. They should be used to make sure that you are confident with the material, and are of approximately the same level of difficulty as the unsolved questions. They also include worked solutions, unlike the unsolved questions section.

1. Find $p_2(x)$ centered around $a = 0$ of e^{2x}, and determine the error of this estimation on the interval $[-2, 1]$.
2. Find the power series for $\frac{1}{(1-x)^2}$.
3. Evaluate
$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{2n} n!}$$

Practice Problem Solutions

1.

Solution: We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \dots.$$

So,

$$e^{2x} = 1 + 2x + \frac{2^2x^2}{2!} + \dots.$$

Meaning that $p_2(x) = 1+2x+2x^2$, since this is the polynomial of degree 2 which best estimates the power series.
We then need to find something which is bigger than (but not that much bigger than)

$$\frac{M(x-a)^{2+1}}{2+1)!} = \frac{M|x-a|^3}{3!}$$

Next, we need to find $f'''(x)$. Since $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$, so $f'''(x) = 8e^{2x}$. Next, we want to see what the largest value of $|f'''(x)|$ is on the interval $[-2, 1]$. Since $8e^{2x}$ is strictly increasing (as its derivative is $16e^{2x}$, which is never negative), we only need to test the values at the endpoints to see what the largest value is. So, note that

$$8e^{2\cdot(-2)} = 8e^{-4} < 8e^2 = 8e^{2\cdot 1}.$$

So, $8e^2$ is the largest value that $|f'''(x)|$ attains, so $M = 8e^2$. Additionally, we know that $x - a = x$, since $a = 0$. So,

$$-2 \leq x - a \leq 1,$$

meaning that $|x - a| \leq 2$. So, $|x - a|^3 \leq 2^3 = 8$. This gives us the final error bound:

$$|f(x) - p_2(x)| \leq \frac{8e^2 \cdot 8}{3!} = \frac{64e^2}{6}.$$

2.

Solution: To find the power series of $\frac{1}{(1-x)^2}$, we note that

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}.$$

So we have that

$$\left(1+x+x^2+x^3+\dots\right)' = \frac{1}{(1-x)^2}.$$

Evaluating the derivative gives us that

$$1+2x+3x^2+4x^3+\dots = \frac{1}{(1-x)^2},$$

which is our final answer.

3.

Solution: The first step to solve any of these questions is to identify if there is a factorial in the power series, and to see what that factorial is. Since we have an $n!$, we are going to match this with the power series for e^x. That is, we need to find a value of "?" such that

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{2n} n!} = \sum_{n=0}^{\infty} \frac{(?)^n}{n!}.$$

We do this by rearranging. The first thing we do is get everything that isn't the factorial as a power of n, or pulled out of the series. Since $2^{2n} = 4^n$, this gives us

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{2n} n!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{4^n n!}.$$

Next, we visually separate the powers of n from everything else. This makes it easier to see how to combine them into one power. This gives us

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{4^n n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{4^n} \frac{1}{n!}.$$

Grouping our powers, we have that our original sum is equal to

$$\sum_{n=0}^{\infty} \left(\frac{(-1)3}{4}\right)^n \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{\left(-\frac{3}{4}\right)^n}{n!} = e^{-\frac{3}{4}}.$$

So,

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{2n} n!} = e^{-\frac{3}{4}}.$$

Unsolved Questions

As I am no longer allowed to distribute practice exams, here is a list of 22 unsolved questions which I believe are of similar difficulty to what might be asked of you on an exam. They are grouped by type of problem, so if you feel like you don't know how to do one specific type of problem, you should do all of the problems of that type for practice.

1. Find the Taylor series centered at $a = -1$ of $f(x) = e^x$.

2. Find the Taylor series centered at $a = \pi$ of $f(x) = \cos(x)$.

3. Find the interval of convergence of
$$\sum_{n=0}^{\infty} \frac{x^n}{n}$$

4. Find the interval of convergence of
$$\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

5. Find the interval of convergence of
$$\sum_{n=0}^{\infty} \frac{(x-5)^n}{3^n}$$

6. Find the interval of convergence of
$$\sum_{n=0}^{\infty} \frac{x^n}{n^n}$$

7. Find the power series of $\ln(1 + 2x + x^2)$.

8. Find the power series of $\tan^{-1}(x)$.

9. Find the power series of $\frac{1}{(1-x)^3}$.

10. Find the power series of $\frac{x}{(1-x)^2}$.

11. Evaluate
$$\lim_{x \to 0} \frac{-1 - x + e^x}{\cos(x) - 1}$$

12. Evaluate
$$\lim_{x \to 0} \frac{x - \sin(x)}{1 - \frac{1}{1-x^3}}$$

13. Determine the value of
$$\sum_{n=0}^{\infty} \frac{2^n \pi^n}{n!}$$

14. Determine the value of

$$\sum_{n=0}^{\infty}(-1)^n \frac{4^n \pi^{2n}}{(2n)!3^{2n}}$$

15. Determine the value of

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{(2n+1)!}$$

16. Determine the value of

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{9^n(2n)!}$$

17. If $f(x) = 3\ln(1+x)$, find $p_2(x)$ centered at $a = 0$ and find the error of the approximation on the interval $\left[-\frac{1}{2}, -\frac{1}{2}\right]$.

18. If $f(x) = e^{3x}$, find $p_4(x)$ centered at $a = 0$ and find the error of the approximation on the interval $[-1, 2]$.

19. If $z = -2 + 2\sqrt{3}$, find the polar form of z.

20. If $z = 3e^{\frac{\pi}{4}i}$, find the standard form of z.

21. Approximate

$$\int_0^{0.2} e^{x^2}\, dx$$

within $\frac{0.2^5}{12}$.

22. Approximate

$$\int_0^{0.3} \frac{\sin(x)}{x}\, dx$$

within $\frac{0.3^5}{120}$.